Princeton, November 15, 1959

Dear Grothendieck,

First of all, a surprising piece of news: Dwork phoned Tate the evening of the day before yesterday to say he had proved the rationality of zeta functions (in the most general case: arbitrary singularities). He did not say how he did it (Karin took the call, not Tate), but I had already seen a manuscript of 1
his in which he proves a substantially weaker result: his method consists in assuming that the variety is a hypersurface in affine space (every variety being birationally isomorphic to such a hypersurface, so that an easy dévissage lets us to pass from this to the general case); in this case he does a computation using "Gauss sums" analogous to Weil's computation for an equation $\sum a_i x_i^{n_i} = b$. Of course, Weil himself had tried to extend his method, but with no success; Delsarte also worked on it; it is thus rather surprising that Dwork was able to do it. Let us wait for confirmation! 2

On your side of things, how is rationality for arbitrary singularities shaping up? Do you think you can get it from your stuff? Is there still a link with homology, and what is it?

Coming back to your projective morphisms and the elimination of the Noetherian conditions: I have now actually checked everything that I claimed in my letter of the day before yesterday. In other words, if $X \rightarrow Y$ is a projective morphism (Y an arbitrary prescheme), there is always a quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ generated by a finitely generated $\mathscr{S}_1$, and such that $X = \mathrm{Proj}(\mathscr{S})$. Thus, it would be worthwhile to define projective morphisms this way!

The proof comes down to checking that part (ii) of Prop 21 in §3 (p.II-70) holds assuming only that S is generated by a finite number of elements of degree 1; this is indeed the case. More generally, under these conditions, if M is a graded S-module and F is a quasi-coherent subsheaf of $\widetilde{M}$, then the submodule N of M made of the guys that locally "belong" to F is such that $\widetilde{N} = F$; the proof is straightforward and very simple.

Once this is done, one feels much more comfortable. In part (ii) of th.3, p.II-97, you can replace "Z Noetherian" by "Z quasi-compact". In th.4, p.II-98, Princeton, November 15, 1959

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